

## Gluon scattering amplitudes at strong coupling

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**ABSTRACT:** We describe how to compute planar gluon scattering amplitudes at strong coupling in  $\mathcal{N} = 4$  super Yang Mills by using the gauge/string duality. The computation boils down to finding a certain classical string configuration whose boundary conditions are determined by the gluon momenta. The results are infrared divergent. We introduce the gravity version of dimensional regularization to define finite quantities. The leading and subleading IR divergencies are characterized by two functions of the coupling that we compute at strong coupling. We compute also the full finite form for the four point amplitude and we find agreement with a recent ansatz by Bern, Dixon and Smirnov.

**KEYWORDS:** Strong Coupling Expansion, Supersymmetric gauge theory, AdS-CFT Correspondence.

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## 1. Introduction

In this article we describe a method for computing gluon scattering amplitudes at strong coupling in  $\mathcal{N} = 4$  super Yang Mills. Of course, these amplitudes are infrared divergent and are not good observables. Nevertheless, in practical computations for collider physics it is often useful to compute these amplitudes as an intermediate step towards computing actual well defined observables. Proper observables are IR finite (IR safe), see *e.g.* [1–3]. Furthermore, in QCD computations it has proven useful to know the  $\mathcal{N} = 4$  super Yang Mills result as a building block [4]. Thus, it is interesting to understand the behavior of scattering amplitudes at strong coupling. We perform the strong coupling computation by using the gauge theory/gravity duality that relates  $\mathcal{N} = 4$  super Yang Mills to string theory on  $AdS_5 \times S^5$  [5]. We consider planar amplitudes. On the string theory side, the leading order result at strong coupling is given by a single classical string configuration associated to the scattering process. A similar result was found by Gross and Mende [6] in their investigation of fixed angle, high energy scattering of strings in flat space. As we explain below, the string theory scattering in AdS is happening at fixed angles and large energy and it is thus determined by a classical solution. The final form

for the color ordered planar scattering amplitude of  $n$  gluons at strong coupling is of the form

$$\mathcal{A} \sim e^{iS_{cl}} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})_{cl}} \tag{1.1}$$

where  $S_{cl}$  denotes the classical action of a classical solution of the string worldsheet equations, which is proportional to the area of the string world-sheet. The solution depends on the momenta,  $k_i^\mu$ , of the gluons. The whole dependence of the coupling is in the overall factor. Of course, we expect  $1/\sqrt{\lambda}$  corrections which we do not compute. The structure of the IR divergences is precisely as expected in the field theory. This comparison enables us to compute the strong coupling expression for the function  $\mathcal{G}_0(\lambda)$  [7–10] characterizing the subleading IR divergent terms. The IR regularization is done via dimensional regularization, as in the field theory. This is achieved by using the gauge theory/gravity duality for Dp-branes for general  $p$  and then performing an analytic continuation in  $p = 3 - 2\epsilon$ .

One of the motivations of this work was the very interesting conjecture by Bern, Dixon and Smirnov [7] (see also [11]) for the all order form of the  $n$  gluon MHV scattering amplitudes. We have computed explicitly the full strong coupling answer for the four point amplitude and found precise agreement with their conjecture. Their conjecture is that the four point amplitude has the form

$$\mathcal{A}_4 = \mathcal{A}_4^{\text{tree}} \exp \left[ (\text{IR divergent}) + \frac{f(\lambda)}{8} (\log(s/t))^2 + (\text{constant}) \right] \tag{1.2}$$

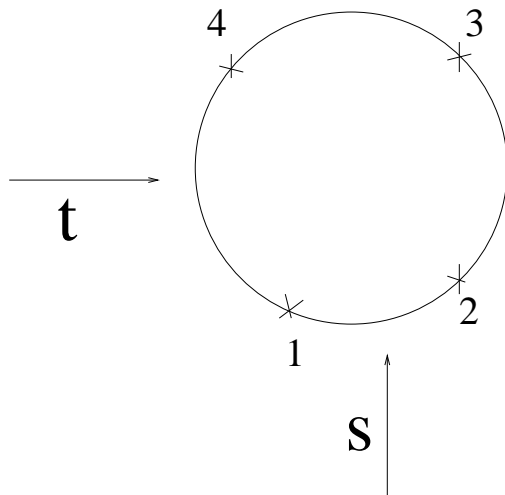
where  $s, t$  are Mandelstam variables,  $f(\lambda)$  is directly related with the cusp anomalous dimension and the IR divergent terms are well characterized Sudakov-like factors [7].

Scattering amplitudes via the *AdS/CFT* duality have been studied in many articles. See for example [12–17] and references therein.

This paper is organized as follows. In section 2 we motivate and describe the general prescription for computing the leading order approximation for the amplitudes. In section 3 we compute explicitly the classical string solution that describes the four point amplitude, then we proceed with a discussion of the structure of infrared divergencies and perform the comparison with the results in [7]. Some conclusions and open problems are sketched in section 4. Many additional remarks and technical details are included in several appendices.

## 2. Gluon scattering amplitudes

Since scattering amplitudes of colored objects are not well defined in the conformal theory it is necessary to introduce an infrared regulator. The answer we obtain will depend on the regularization scheme. Once we compute a well defined (IR safe) physical observable the IR regulator will drop out. An example of a well defined physical observable is the amplitude for a process involving narrow jets going in some specific angular directions. The answer will depend on the precise definition of the jet observable, but not on the IR regulator. Further discussion of these issues can be found in [1–3]. A popular regularization scheme is dimensional regularization and we will use it in the next section. For the time being, it is convenient to use a different IR regulator. In terms of the gravity dual, this IR regulator



**Figure 1:** Order of the particles in the diagram and definition of  $s$  and  $t$  for the four point amplitude.

is a D-brane that extends along the worldvolume directions but is localized in the radial direction. In other words, we start with the  $AdS_5$  metric

$$ds^2 = R^2 \left[ \frac{dx_{3+1}^2 + dz^2}{z^2} \right] \quad (2.1)$$

and we place a D-brane at a large value  $z_{IR}$ . In terms of the field theory, such D-branes arise, for example, if we go to the Coulomb branch of the theory. The asymptotic states are open strings that end on the D-brane<sup>1</sup>. We then scatter these open strings. We are interested in keeping the momentum fixed as we take away the IR cutoff. This means that the proper momentum of the strings living at  $z_{IR}$  is very large,  $kz_{IR} \gg 1$ , where  $k$  is the momentum in field theory units ( $k$  is conjugate to translations in  $x$ ). Thus, we are studying the scattering of open strings at fixed angles and very high momentum. Such amplitudes were studied in flat space by Gross and Mende, [6]. The important feature noted in [6], is that amplitudes at high momentum transfer are dominated by a saddle point of the classical action.<sup>2</sup> Thus, in order to compute the amplitude we simply have to compute a solution of the classical action. In our case we need to consider a classical string in AdS.

Let us first recall the flat space case [6]. We consider scattering of open strings on a  $D$  brane that is transverse to the direction  $z$ , sitting at  $z = z_{IR}$ . We want to consider a worldsheet with the topology of disk with vertex operator insertions on its boundary. See

<sup>1</sup>The color structure of the amplitude is not very apparent with this regularization. We can get a glimpse of it by introducing  $N$  branes in the IR (instead of just one). In other words, we go to a generic point in the Coulomb branch. Then the external gluons are strings stretching between these branes. The color ordered amplitude would arise when we consider a configuration where one open string ends on the brane where the next open string starts. Namely, the  $i$ th open string goes between branes  $i$  and  $i + 1$ .

<sup>2</sup>Here we concentrate on genus zero amplitude, [6] computed also higher genus amplitudes using the same idea.

figure 1. Near each vertex operator insertion the solution should behave as <sup>3</sup>

$$x^\mu \sim i k^\mu \log |w|^2 \tag{2.2}$$

where  $k^\mu$  is the momentum of the open string. In addition we require  $z = z_{\text{IR}}$  on the boundaries of the worldsheet. The solution is

$$x^\mu = i \sum_i k_i^\mu \log |w - w_i|^2, \quad z = z_{\text{IR}} \tag{2.3}$$

Actually, we still need to determine  $w_i$ . The actual solution that minimizes the action has fixed values of  $w_i$ , up to conformal transformations on the worldsheet. For example, in the case of the four point function we can set three of the  $w_i$  to  $w_1 = 0$ ,  $w_3 = 1$ ,  $w_4 = \infty$  and then  $w_2$  is determined by inserting the above solution into the action and minimizing with respect to  $w_2$  which gives

$$w_2 = \frac{s}{s+t}; \quad s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_4)^2 \tag{2.4}$$

The value of the action is then

$$S_{\text{flat}}(s, t) = s \log(-s) + t \log(-t) - (s+t) \log(-s-t) \tag{2.5}$$

and the leading approximation to the scattering amplitude is then  $\mathcal{A} = e^{-S}$ . Notice that the worldsheet has Euclidean signature in the regime that  $s, t < 0$  (spacelike  $s$  and  $t$  channel momentum transfer, and timelike  $u$  channel momentum transfer). In this regime the solution (2.3) is such that the coordinates along the brane,  $x^\mu$ , are purely imaginary. This amplitude is exponentially suppressed and it represents the very small probability process where the two incoming string states tunnel to the two outgoing ones. Notice that we have an open string amplitude and the order of the boundary vertex operators is important. Note that this leading exponential behavior (2.5) is independent of the particular string states we are scattering, as long as we keep them fixed when we increase the momentum transfer.

Let us now consider the AdS case. Now it will be more difficult to find the classical solution. However, there is one important aspect of the classical solution that we can understand in a qualitative way. Namely, we expect that the solution will be such that in the central region of the collision the value of  $z$  will be much smaller than  $z_{\text{IR}}$  and that it will be roughly proportional to the inverse of  $\sqrt{-s}$  or  $\sqrt{-t}$  since they set the off shell momentum transfer of the process. This expectation is based on the idea that the  $z$  direction should correspond to the off shell energy scale of the process. In [14] a similar scattering process was considered and it was also found that the leading contribution came from  $1/z \sim \sqrt{(-s)}$ .<sup>4</sup>

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<sup>3</sup>We set  $\alpha' = 1$ .

<sup>4</sup>The main difference with our configuration is that in [14] the asymptotic states were closed strings in the bulk. Thus they could move in the  $z$  direction more easily than the open strings we consider, which are attached to a D-brane at  $z_{\text{IR}}$ . The open strings can move in  $z$  by stretching away from the D-brane, which is a stringy excitation.

In order to state most simply the boundary conditions for the worldsheet it is convenient to describe the solution in terms of T-dual coordinates  $y^\mu$  defined in the following way. We start with a metric that contains

$$ds^2 = w^2(z)dx_\mu dx^\mu + \dots \tag{2.6}$$

where  $w$  is the warp factor. We define T-dual variables  $y^\mu$  by

$$\partial_\alpha y^\mu = iw^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu \tag{2.7}$$

In the regime under consideration the T-dual coordinates are real and the worldsheet is Euclidean. In addition, the boundary condition for the original coordinates  $x^\mu$ , which is that they carry momentum  $k^\mu$ , translates into the condition that  $y^\mu$  has “winding”

$$\Delta y^\mu = 2\pi k^\mu \tag{2.8}$$

Note that we are not taking the coordinates to be compact (specially time!). One can view this as purely a mathematical operation that makes it easier to find the classical solutions.<sup>5</sup> The T-duality will also produce a non-trivial dilaton field, but it will not affect our classical solutions. The T-dual metric is again  $AdS_5$  after defining  $r = \frac{R^2}{z}$ <sup>6</sup>

$$d\tilde{s}^2 = R^2 \left[ \frac{dy_\mu dy^\mu + dr^2}{r^2} \right], \quad r = \frac{R^2}{z} \tag{2.9}$$

The full Green Schwartz string action was computed in these “T-dual” variables in [18].

The solution we want is a surface which at  $r = R^2/z_{IR}$  ends on a particular one dimensional line which is constructed as follows. For each particle we have a lightlike segment joining two points separated by (2.8). We concatenate these segments according to the ordering of the open strings in the disk diagram. This ordering is interpreted as the particular color ordering of the amplitude. See figure 2. The resulting line consists of lightlike segments. The condition that the line closes corresponds to the momentum conservation condition. As we explained above the solution lives at values of  $r > r_{IR} = R^2/z_{IR}$ . As we take the limit  $z_{IR} \rightarrow \infty$  we find that the boundary of the worldsheet moves to the boundary of the T-dual metric (2.9) which is at  $r = 0$ . From the point of view of the T-dual metric (2.9) the computation that we are doing is formally the same as the one we would do [19] if we were computing (in the classical string approximation) the expectation value of a Wilson loop given by a sequence of lightlike segments.<sup>7</sup>

In conclusion, the leading exponential behavior of the  $n$ -point scattering amplitude is given by the area,  $A$ , of the minimal surface that ends on a sequence of lightlike segments on the boundary of (2.9)

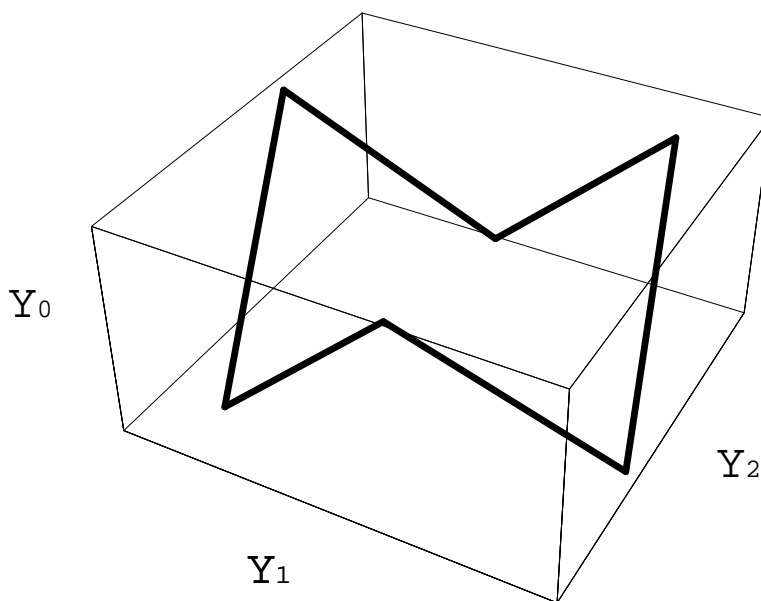
$$\mathcal{A} \sim e^{-\frac{R^2}{2\pi}A} = e^{-\frac{\sqrt{\lambda}}{2\pi}A} \tag{2.10}$$

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<sup>5</sup>One could consider situations with compact spatial coordinates. Then we would be doing an honest T-duality on the spatial coordinates.

<sup>6</sup>Notice that we do not do a T-duality in the  $z$  direction.

<sup>7</sup>This prescription is vaguely reminiscent to the one used for non-commutative theories [20].



**Figure 2:** The kinematic data, namely the sequence of momenta  $k_1^\mu, \dots, k_n^\mu$ , translates into a sequence of lightlike segments joining points in the T-dual space parametrized by  $y^\mu$ . These points are separated by  $2\pi k_i^\mu$ . This curve lives at  $r = 0$  in the T-dual AdS space (2.9). We need to find a minimal surface that ends on this curve.

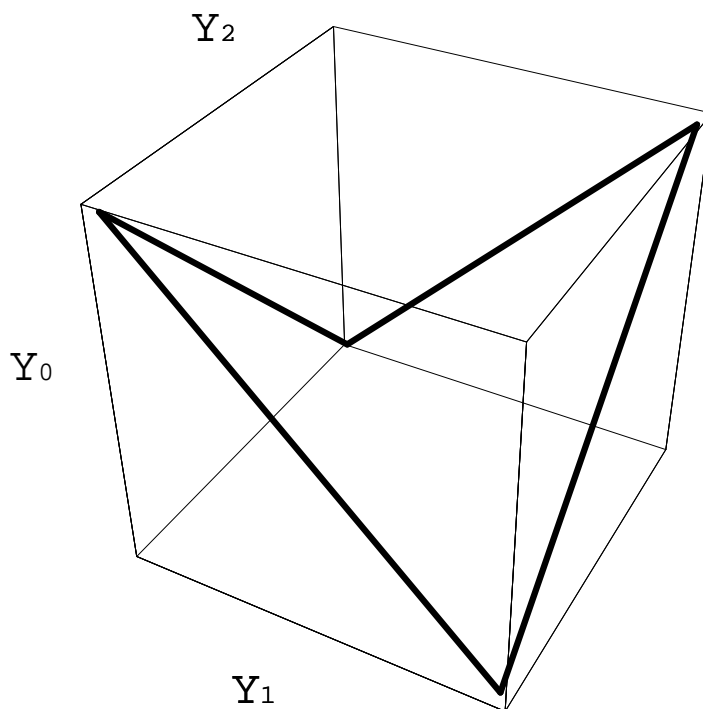
The area  $A = A(k_1^\mu, \dots, k_n^\mu)$  contains the kinematic information about the momenta. The amplitude is color ordered in a way that is reflected by the particular order in which the lightlike segments are arranged along the boundary. There is no information about the particular polarizations or states of the gluons, which contribute to prefactors in (2.10), and are of subleading order in  $1/\sqrt{\lambda}$ . From the worldsheet point of view, we will need to supply this information once we quantize around the classical solution. Such terms are beyond the scope of this paper. Note that the  $\lambda$  dependence is contained purely in the factor multiplying the area.<sup>8</sup>

The result (2.10) is still somewhat formal since the area is infinite due to the infrared divergences that we mentioned above. In order to find a finite answer we will need to regularize the result. In the next section we will discuss the structure of the infrared divergences and we will do an explicit computation for the four point amplitude.

A reader familiar with AdS/CFT might be surprised by the fact that in terms of the original coordinates we are setting boundary conditions at  $r = 0$  or  $z = \infty$ , rather than at  $z = 0$ . This apparent confusion goes away once we consider AdS in global coordinates, which is better in order to see whether we are at the boundary or not. In those coordinates one can see that the surface  $z = \infty$  indeed intersects the boundary, as  $x^\mu \rightarrow \infty$  and we will later check explicitly that the solution in terms of the coordinates  $(x^\mu, z)$  is intersecting the boundary of AdS. This is discussed in more detail in appendix A. Thus, there is no

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<sup>8</sup>The area is computed in an AdS space with radius one.



**Figure 3:** Sequence of lightlike segments which specifies the scattering configuration. This figure lives at  $r = 0$  of the metric (2.9).

contradiction with the general principle stating that good observables are defined on the boundary of AdS.

### 3. Computation of the four point amplitude at strong coupling

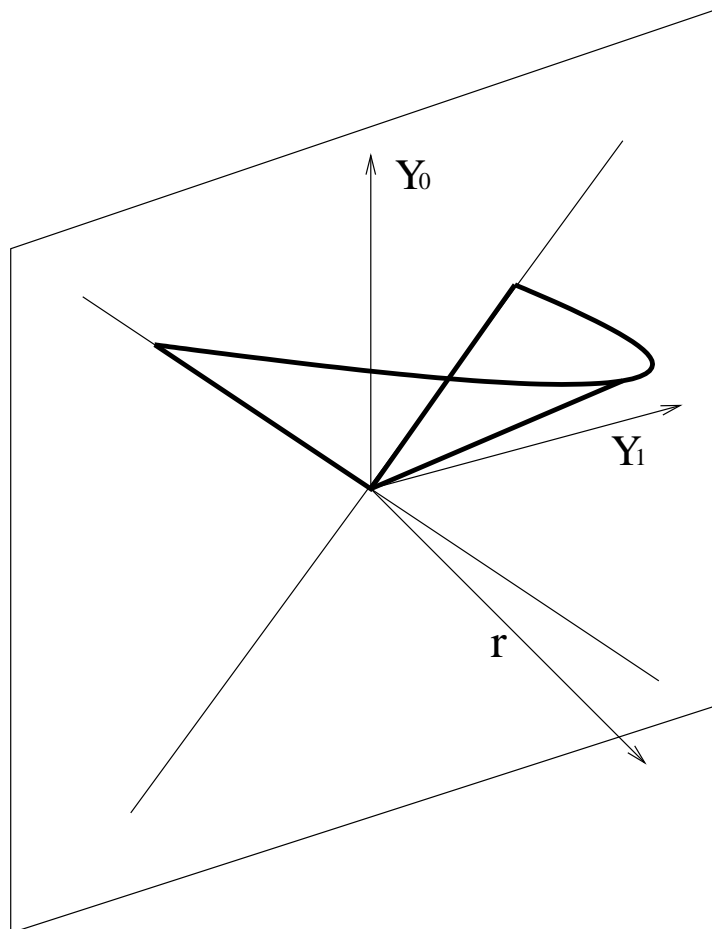
In this section we consider the planar four gluon amplitude. We label the momenta as  $k_1, k_2, k_3, k_4$ , where the subindex indicates the color ordering, see figure 1. We consider the region where particles 1 and 3 are incoming and particles 2 and 4 are outgoing. We label by  $k$  the center of mass energy or momentum of each of the incoming particles and we denote by  $\varphi$  the scattering angle in the center of mass frame. We introduce the usual Mandelstam variables

$$\begin{aligned}
 s &= -(k_1 + k_2)^2 = -2k_1 \cdot k_2 = -4k^2 \sin^2 \frac{\varphi}{2} \\
 t &= -(k_1 + k_4)^2 = -2k_1 \cdot k_4 = -4k^2 \cos^2 \frac{\varphi}{2} \\
 u &= -(k_1 + k_3)^2 = -2k_1 \cdot k_3 = 4k^2 = -(s + t)
 \end{aligned}
 \tag{3.1}$$

We will focus on the region where  $s, t < 0$  (they correspond to spacelike momentum transfer).

As we explained above, we should find a classical string solution specified by these momenta. It is simplest to think about the solution in T-dual coordinates where the





**Figure 4:** The lightlike cusp. The thin lines shows the light cone.

problem boils down to finding a minimal surface in the T-dual  $AdS_5$  space (2.9). This surface ends at  $r = 0$  on a closed sequence of lightlike segments whose sides are specified by the lightlike momentum vectors  $(2\pi)k_i^\mu$ , see figure 3.

### 3.1 The lightlike cusp

We start by considering the solution near the cusp where two of the lightlike lines meet. So we consider two semi infinite lightlike lines meeting at a point. This case was considered in [21] and it will prove useful for generating the solution we want. It is a surface that can be embedded in an  $AdS_3$  subspace of  $AdS_5$

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dr^2}{r^2} \tag{3.2}$$

We are interested in computing the surface ending on a light-like Wilson loop which is along  $y^1 = \pm y^0, y^0 \geq 0$ ,<sup>9</sup> see figure 4. The problem has a boost and scaling symmetry that

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<sup>9</sup>One can also consider Wilson loops along  $y^0 = \pm y^1, y^1 > 0$ , the basic difference with the ones considered here is that their world-sheet is Lorentzian and  $z$  is imaginary.

becomes explicit if we choose the following parametrization

$$y_0 = e^\tau \cosh \sigma, \quad y_1 = e^\tau \sinh \sigma, \quad r = e^\tau w \tag{3.3}$$

Boosts and scaling transformations are simply shifts of  $\sigma$  and  $\tau$ . Then the Nambu-Goto action becomes

$$S = \frac{R^2}{2\pi} \left( \int d\sigma \right) \int d\tau \frac{\sqrt{1 - (w(\tau) + w'(\tau))^2}}{w(\tau)^2} \tag{3.4}$$

One can explicitly check that  $w(\tau) = \sqrt{2}$  solves the equations of motion, hence the surface is given by

$$r = \sqrt{2} \sqrt{y_0^2 - y_1^2} = \sqrt{2} \sqrt{y^+ y^-}, \quad y^\pm = y^0 \pm y^1 \tag{3.5}$$

When we insert the solution in the action (3.4) we find that the lagrangian is purely imaginary, this means that the amplitude  $\mathcal{A} \sim e^{iS}$  will have an exponential suppression factor

$$iS = -S_E = -\frac{R^2}{4\pi} \int d\sigma d\tau \tag{3.6}$$

where  $S$  is the action for a spacelike surface embedded in the Lorentzian target space that we are considering. This integral is infinite. We will later discuss its regularization.

It is instructive to study this solution in terms of embedding coordinates. These are coordinates where we view  $AdS_5$  as the following surface embedded in  $R^{2,4}$

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1 \tag{3.7}$$

The relation between these and the Poincare coordinates in (2.9) is

$$\begin{aligned} Y^\mu &= \frac{y^\mu}{r}, \quad \mu = 0, \dots, 3 \\ Y_{-1} + Y_4 &= \frac{1}{r}, \\ Y_{-1} - Y_4 &= \frac{r^2 + y_\mu y^\mu}{r} \end{aligned} \tag{3.8}$$

We can now write the surface corresponding to the cusp in terms of the equations [21]

$$Y_0^2 - Y_{-1}^2 = Y_1^2 - Y_4^2, \quad Y_2 = Y_3 = 0 \tag{3.9}$$

### 3.2 The four lightlike segments solution

We now consider a Wilson loop containing four light-like edges, which contains four cusps like the one considered above. The configuration of lightlike lines is shown in figure 3.

In order to write the Nambu-Goto action it is convenient to consider Poincare coordinates  $(r, y_0, y_1, y_2)$ , setting  $y_3 = 0$ , and parametrize the surface by its projection to the  $(y_1, y_2)$  plane. We consider first the case with  $s = t$  where the projection of the Wilson lines in figure 3 is a square. The Nambu-Goto action is then the action for two fields  $y_0$  and  $r$  living on a square parametrized by  $y_1$  and  $y_2$ . The action reads

$$S = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2} \tag{3.10}$$

By scale invariance, we can change the size of the square. We choose the edges of the square to be at  $y_1, y_2 = \pm 1$ . The boundary conditions are then given by

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1 \quad (3.11)$$

From the solution for the single cusp, we can obtain, after boost transformations, the form of the solution in the vicinity of any of the cusps. The following expression for the fields can be easily seen to have the right behavior close to the cusps

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)} \quad (3.12)$$

Remarkably it turns out to be a solution of the equations of motion. When expressed in terms of embedding coordinates, the surface is given by the equations

$$Y_3 = 0, \quad Y_4 = 0, \quad Y_0 Y_{-1} = Y_1 Y_2 \quad (3.13)$$

In fact this solution is related by the AdS isometries (SO(2, 4) transformations) to the cusp solution (3.9)<sup>10</sup>. The reader might be puzzled by the following. We seem to have mapped a solution with two lightlike lines on the boundary to one with four lightlike lines. The AdS isometries are conformal transformations on the boundary and conformal transformations preserve angles and cannot produce cusps where there were none. The solution to this apparent puzzle is that the cusp solution (3.9) really has four cusps once it is embedded in global coordinates [21]. We miss some of the cusps when we use Poincare coordinates because those cusps are on the boundary of the Minkowski space parametrized by  $y^\mu$ . In fact this is a simpler alternative way to derive the solution. Namely we start with the cusp solution (3.9), notice that it really has four cusps and then map it through conformal transformations to the solution we really want (3.12) -(3.13).

After we understood this point it becomes a simple exercise to compute the solution for general  $s$  and  $t$ . We simply need to apply an SO(2, 4) transformation to the solution we already have. Starting from (3.13) we perform a boost in the 04 plane and obtain

$$Y_4 - v Y_0 = 0, \quad Y_{-1} \gamma (Y_0 - v Y_4) = \gamma^{-1} Y_0 Y_{-1} = Y_1 Y_2, \quad Y_3 = 0 \quad (3.14)$$

where  $\gamma^{-1} = \sqrt{1 - v^2}$ .

Let us now write the solutions in terms of worldsheet coordinates in conformal gauge. Let us first go back to the solution for the case with  $s = t$ , (3.12), and compute the induced metric on the worldsheet. We find

$$ds^2 = \frac{dy_1^2}{(1 - y_1^2)^2} + \frac{dy_2^2}{(1 - y_2^2)^2} = du_1^2 + du_2^2, \quad \text{where} \quad y_i = \tanh u_i \quad (3.15)$$

Notice that the metric on the worldsheet is Euclidean. More precisely, we have a spacelike surface embedded in a Lorentzian target space. Written in terms of  $u_i$  coordinates

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<sup>10</sup>In order to go from (3.9) to (3.13) we take  $Y_2 \rightarrow Y_4$ ,  $Y_0 \rightarrow \frac{1}{\sqrt{2}}(Y_0 + Y_{-1})$ ,  $Y_{-1} \rightarrow \frac{1}{\sqrt{2}}(Y_0 - Y_{-1})$ ,  $Y_1 \rightarrow \frac{1}{\sqrt{2}}(Y_1 + Y_2)$  and  $Y_4 \rightarrow \frac{1}{\sqrt{2}}(Y_1 - Y_2)$

the solution (3.12) becomes

$$\begin{aligned}
 y_1 &= \tanh u_1, & y_2 &= \tanh u_2, & r &= \frac{1}{\cosh u_1 \cosh u_2}, & y_0 &= \tanh u_1 \tanh u_2 \\
 Y_0 &= \sinh u_1 \sinh u_2, & Y_1 &= \sinh u_1 \cosh u_2, & Y_2 &= \cosh u_1 \sinh u_2, \\
 Y_{-1} &= \cosh u_1 \cosh u_2, & Y_4 &= Y_3 = 0
 \end{aligned} \tag{3.16}$$

This is now a solution of the equations in conformal gauge, whose action reads

$$iS = -\frac{R^2}{2\pi} \int \mathcal{L} = -\frac{R^2}{2\pi} \int du_1 du_2 \frac{1}{2} \frac{(\partial r \partial r + \partial y_\mu \partial y^\mu)}{r^2} \tag{3.17}$$

Note that the metric (3.15) is Euclidean, this is responsible for the extra  $i$  in this formula. The lagrangian density evaluated on the solution is simply  $\mathcal{L} = 1$ . Performing the boost (3.14) and a simple rescaling we now find the solution for  $s \neq t$

$$\begin{aligned}
 r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \\
 y_0 &= \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\
 y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \\
 y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}
 \end{aligned} \tag{3.18}$$

where  $b = v\gamma$  and we consider  $b < 1$ . The parameter  $a$  sets the overall scale of the momentum. The solution approaches the boundary of  $AdS_5$  where  $u_1$  or  $u_2$  go to plus or minus infinity. These four possibilities correspond to the four lightlike lines on the boundary. For example, if we take  $u_1 \rightarrow +\infty$  we find that  $r = 0$  and

$$y_1 = \frac{a}{1 + b \tanh u_2}, \quad y_2 = \frac{a \tanh u_2}{1 + b \tanh u_2}, \quad y_0 = a\sqrt{1+b^2} \frac{\tanh u_2}{1 + b \tanh u_2} \tag{3.19}$$

We see that  $y_1 + by_2 = a$  and that we have a lightlike line going between two points whose projections on the  $y_1, y_2$  plane are located at

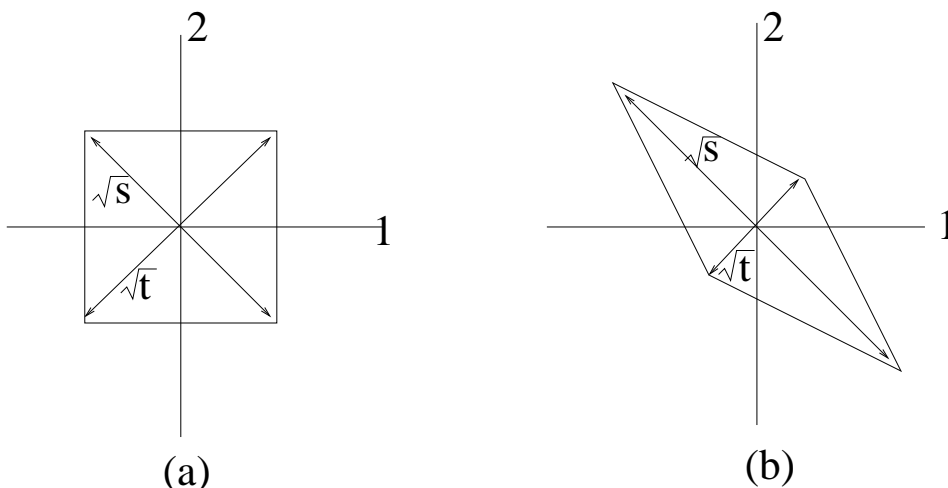
$$\text{A : } y_1 = y_2 = \frac{a}{1+b}, \quad \text{and} \quad \text{B : } y_1 = -y_2 = \frac{a}{1-b} \tag{3.20}$$

which are reached at  $u_2 \rightarrow \pm\infty$ . By considering other limits we get the other segments. The values of  $s$  and  $t$  are given by the square of the distance between the vertex of two non-adjacent cusps, see figure 5. In terms of the parameters  $a$  and  $b$  they are given by

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \quad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}, \quad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2} \tag{3.21}$$

where the factors of  $2\pi$  comes from (2.8).

The solution and the value of the action are symmetric under  $s \leftrightarrow t$ , which is a symmetry of the full problem.



**Figure 5:** Projection of the light like lines on the  $y_1, y_2$  plane for (a)  $s = t$  and (b)  $s \neq t$ . The line also moves in the time direction with a slope such that we get a lightlike line. Points on opposite vertices sit at equal times. Time goes up and down as we move from segment to segment.

It is also instructive to write the solution in terms of the original AdS coordinates (2.1). We obtain

$$\begin{aligned}
 x_1 &= \frac{iR^2}{a} \left( \frac{u_2}{2} + \frac{1}{4} \sinh 2u_2 + b \left( -\frac{u_1}{2} + \frac{1}{4} \sinh 2u_1 \right) \right), \\
 x_2 &= \frac{iR^2}{a} \left( -\frac{u_1}{2} - \frac{1}{4} \sinh 2u_1 + b \left( \frac{u_2}{2} - \frac{1}{4} \sinh 2u_2 \right) \right) \\
 x_0 &= \frac{iR^2}{2a} \sqrt{1 + b^2} (\cosh^2 u_2 - \cosh^2 u_1), \\
 z &= \frac{R^2}{a} (\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2), \tag{3.22}
 \end{aligned}$$

We see that for large  $u_i$  the solution is such that it carries momentum in the spatial directions. The solution lives in complexified AdS space since it represents a sort of tunnelling solution. For  $u_i \rightarrow \pm\infty$  the solution goes to the region where  $z$  is large, which naively seems to correspond to the IR of the field theory. On the other hand, we are also finding that  $x^\mu$  are going to infinity at the same time. We can go to global coordinates and find that the solution (3.22) indeed approaches the boundary of AdS. It touches the boundary on the surface with  $1/z = 0$  which corresponds to the region of the boundary that is the boundary of the Penrose diagram of the Minkowski slices parametrized by  $x^\mu$ . See appendix A. In the central scattering region,  $u_i \sim 0$ , we have that  $z \sim R^2/a \sim R^2(\frac{1}{\sqrt{-s}} + \frac{1}{\sqrt{-t}})$  which is in agreement with the discussion in section two, where we said that the scale of momentum transfer should set the minimum value of  $z$  that the solution explores.<sup>11</sup>

<sup>11</sup>The above behavior can be qualitatively understood as the result of two competing factors. On one side, high proper momentum transfers suppress the scattering and this suppression can be lowered by going to regions of smaller  $z$ . On the other hand we have to pay a price for stretching the string away from  $z = z_{\text{IR}}$  in the  $z$  direction

We are now almost ready to evaluate the action. However, we should note that there is a small subtlety. The action in terms of the original coordinates and the action in terms of the T-dual coordinates differ by a total derivative which will contribute with a boundary term. The correct action to evaluate turns out to be the area of the surface in the T-dual coordinates. When we consider the problem in the original coordinates we should remember that we are putting boundary conditions that fix the momentum at each boundary. In order to have a proper variational principle we should add a boundary term. This boundary term is precisely the one that turns the original action into the T-dual action.

### 3.3 Dimensional regularization in the gravity dual

As we mentioned above the action of the solutions we have discussed is infinite. A popular regularization method for  $\mathcal{N} = 4$  super Yang Mills is the so called dimensional reduction scheme [22]. In this scheme one goes to a general dimension  $D = 4 - 2\epsilon$  but one continues to use a theory with 16 supercharges. In other words, one considers the dimensional reduction of ten dimensional super Yang Mills to  $4 - 2\epsilon$  dimensions. For integer dimensions these are precisely the low energy theories living on Dp branes, where  $p = D - 1$ . The gravity dual of these theories involves the string frame metric [23]

$$\begin{aligned} ds^2 &= f^{-1/2} dx_D^2 + f^{1/2} [dr^2 + r^2 d\Omega_{9-D}^2], \\ D &= 4 - 2\epsilon f = \frac{c_D \lambda_D}{r^{8-D}}, \\ c_D &= 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2 + \epsilon) \end{aligned} \tag{3.23}$$

where  $\lambda_D = g_D^2 N$  and  $g_D^2$  is the coupling<sup>12</sup>. We parametrize the coupling in  $D$  dimensions in terms of the IR cutoff scale  $\mu$  as it was done in the field theory analysis of [7]

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^\epsilon}, \quad \gamma = -\Gamma'(1) \tag{3.24}$$

where  $\lambda = \lambda_4$  is the dimensionless four dimensional coupling which is kept fixed as we vary  $\epsilon$ .

From now on, we will drop the sphere part of the metric since it does not play an important role. We now compute the metric for the T-dual variables defined through (2.7) (with  $w^2 = f^{-1/2}$ ). We get

$$ds^2 = f^{1/2} (dy_D^2 + dr^2) = \sqrt{c_D \lambda_D} \left( \frac{dy_D^2 + dr^2}{r^{2+\epsilon}} \right) \tag{3.25}$$

Notice that the IR region of the original metric corresponds to the region where  $r \sim 0$  in the T-dual metric (3.25).

In the region very close to  $r \sim 0$  we cannot trust the gravity dual and we should use the weakly coupled field theory in the IR. In this field theory picture we can define the asymptotic gluon states. We will see that we will only need the geometry in the region where we can trust it if we are interested in the strong coupling behavior.

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<sup>12</sup>This is defined as the coefficient of the  $D$  dimensional action in the usual way  $S = \frac{1}{4g^2} Tr[F^2]$  and  $Tr[T^a T^b] = \frac{1}{2} \delta_{ab}$  where  $T^a$  are the generators of SU(N).

### 3.4 Evaluation of the action in dimensional regularization

We regularize the theory by considering  $Dp$ -branes, with  $p = 3 - 2\epsilon$ . We are then lead to the following action

$$S = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon} \tag{3.26}$$

where  $\mathcal{L}_{\epsilon=0}$  is the lagrangian density for  $AdS_5$ , as in (3.10) or (3.17).

In order to understand how dimensional regularization works, let us perform the computation for the lightlike cusp in general dimensions. We still have the boost symmetry and we can make an ansatz similar to the one in (3.3) but now the Lagrangian depends explicitly on  $\tau$

$$S = \frac{\sqrt{c_D \lambda_D}}{2\pi} \int d\sigma \int d\tau e^{-\epsilon\tau} \frac{\sqrt{1 - (w + w')^2}}{w^{2+\epsilon}} \tag{3.27}$$

It turns out that a constant  $w$  is still a solution, but now the constant is

$$w = \sqrt{2} \sqrt{1 + \frac{\epsilon}{2}} \quad \rightarrow \quad r = \sqrt{2} \sqrt{1 + \epsilon/2} \sqrt{y^+ y^-} \tag{3.28}$$

Inserting this solution into the action and writing the integral as an integral over  $y^\pm$  we get

$$-iS = A_\epsilon \int \frac{dy^+ dy^-}{(2y^+ y^-)^{1+\epsilon/2}} = \frac{4}{\epsilon^2} \frac{A_\epsilon}{(2y_c^+ y_c^-)^{\epsilon/2}}, \quad A_\epsilon = \frac{\sqrt{c_D \lambda_D} \sqrt{1 + \epsilon}}{8\pi(1 + \epsilon/2)^{1+\epsilon/2}} \tag{3.29}$$

where  $y_c^\pm$  is a cutoff for large  $y_c^\pm$ .<sup>13</sup> We will later see that this double pole is in agreement with field theory expectations. One might be worried that the metric (3.25) is getting highly curved as  $r \rightarrow 0$  (for  $\epsilon < 0$ ). Fortunately this is not a problem if we are only interested in determining the strong coupling behavior of the amplitude. We see this as follows. The integral is cutoff at the point where

$$r^\epsilon \sim 1 \tag{3.30}$$

But the effective curvature,  $\mathcal{R}$ , at that point is of the form

$$\mathcal{R} \sim \frac{1}{\sqrt{\lambda}} r^\epsilon \ll 1, \quad \text{if} \quad \lambda \gg 1 \tag{3.31}$$

Thus, for large lambda we can perform dimensional regularization without worrying about the region with strong curvature. If we wanted to understand the result at all values of  $\lambda$ , then it would be important to understand the whole region. A simple way to put it, is to say that dimensional regularization and strong coupling commute.

Let us now turn to the problem of the four-point scattering amplitude. When  $\epsilon \neq 0$  we have a different lagrangian (3.26) and we would need to find the solutions for the new lagrangian, as we did above for the cusp. Fortunately, there is a simple trick that allows

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<sup>13</sup>In the computation of the actual amplitudes this will be the momenta of the particles.

us to find the solution to the accuracy that we need.<sup>14</sup> We first note that if we have a lagrangian which has the expansion  $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2$ , then we can expand the solutions of the equations of motion as  $q = q_0 + \epsilon q_1 + \dots$ . We will be interested in evaluating the final answer to zeroth order in  $\epsilon$ . However, since we have IR divergencies we find that the leading order solution  $q_0$  will give a leading double pole in  $\epsilon$ . Thus when we do the formal expansion we mentioned above, we will want to evaluate the action to the formal order  $\epsilon^2$ , which in reality will be order  $\epsilon^0$ . Note that then we will not need to know the solution to second order in  $\epsilon$  since that will contribute a term of the form  $\int \frac{\partial \mathcal{L}_0}{\partial q} |_{q_0} q_2$  which vanishes due to the fact that  $q_0$  obeys the zeroth order equations of motion. For a similar reason the first order solution  $q_1$  will only contribute to terms of formal order  $\epsilon^2$ . Thus to real orders  $\epsilon^{-2}$  and  $\epsilon^{-1}$  we can simply evaluate the zeroth order solution in the new,  $\epsilon$  dependent lagrangian and we will get an accurate enough answer. Since the first order solution can only contribute to formal order  $\epsilon^2$  we will need an IR divergence of order  $1/\epsilon^2$  to give a finite answer. These divergences only arise in the cusps. Thus, we only need the  $q_1$  solution near the cusps. However, for the cusp region we know the solution (3.28) When we have the general zeroth order solution of the problem, we can get a solution that is accurate enough at the cusps by writing

$$r_\epsilon \sim \sqrt{1 + \frac{\epsilon}{2}} r_{\epsilon=0}, \quad y_\epsilon^\mu \sim y_{\epsilon=0}^\mu \tag{3.32}$$

where the  $\epsilon = 0$  solutions are the ones we discussed above. Inserting these expressions into the action (3.26) gives us an accurate enough answer to extract the finite pieces in the amplitude. A more detailed analysis of the finite  $\epsilon$  equations in the various regions (near the cusps, near the lines) shows that the above argument is indeed correct.

By inserting these expression into the action we get

$$-iS = B_\epsilon \int_{-\infty}^{\infty} du_1 du_2 (\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2)^\epsilon (1 + \epsilon I_1 + \epsilon^2 I_2 + \dots) \tag{3.33}$$

where

$$\begin{aligned} I_1 &= \frac{(b^2 - 1)(\cosh 2u_1 + \cosh 2u_2) - 2(1 + b^2)}{8(\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2)^2} \\ I_2 &= \frac{1 + b^2 - (1 + b^2) \cosh 2u_1 \cosh 2u_2 - 2b \sinh 2u_1 \sinh 2u_2}{16(\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2)^2} \\ B_\epsilon &= \frac{\sqrt{\lambda_D c_D}}{2\pi} \frac{1}{a^\epsilon} \end{aligned} \tag{3.34}$$

where we have expanded up to terms that give finite order answers in the final result. The integrals can be performed as explained in appendix B. The final result is

$$iS = -B_\epsilon \left( \frac{\pi \Gamma[-\frac{\epsilon}{2}]^2}{\Gamma[\frac{1-\epsilon}{2}]^2} {}_2F_1 \left( \frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2 \right) + 1 \right) \tag{3.35}$$

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<sup>14</sup>In principle, one could compute the scattering amplitudes for gluons in the case of  $D = 5, 6$  if the center of mass energy is such that we can trust the corresponding gravity solutions [23]. That would require solving the equations for the new lagrangian (3.26) with  $\epsilon = -\frac{1}{2}, -1$ .



It is then straightforward to expand in powers of  $\epsilon$  up to finite contributions. We need to recall the expressions for  $s, t$  in (3.21) and also the formulas for  $c_D$  (3.23) and  $\lambda_D$  (3.24). Putting all this together we get the final answer

$$\mathcal{A} = e^{iS} = \exp \left[ iS_{\text{div}} + \frac{\sqrt{\lambda}}{8\pi} \left( \log \frac{s}{t} \right)^2 + \tilde{C} \right] \quad (3.36)$$

$$\begin{aligned} \tilde{C} &= -\frac{\sqrt{\lambda}}{4\pi} \left( 1 - \frac{\pi^2}{3} - 2 \log 2 + (\log 2)^2 \right) \\ iS_{\text{div}} &= 2iS_{\text{div},s} + 2iS_{\text{div},t} \end{aligned} \quad (3.37)$$

where  $S_{\text{div},s}$  and  $S_{\text{div},t}$  are the divergent pieces associated to each cusp or pair of consecutive gluons. There are two pairs with total squared momentum  $t$  and two with  $s$ . We have <sup>15</sup>

$$iS_{\text{div},s} = -\frac{1}{\epsilon^2} \frac{1}{2\pi} \sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon} \frac{1}{4\pi} (1 - \log 2) \sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} \quad (3.38)$$

and  $S_{\text{div},t}$  is given by a similar expression with  $s \rightarrow t$ . We now compare what we obtained here with the field theory results and conjectures in [7].

### 3.5 IR divergences of amplitudes

In this section we recall some general results on IR divergences of scattering amplitudes [7–9]. We will focus here on planar diagrams and color ordered amplitudes. The first result is that the IR singularities of the amplitude can be associated to consecutive gluons in the color ordered amplitude. This is fairly clear once we recognize that the IR singularities come from low momentum gluons and we use that we consider only planar diagrams. The leading divergence goes like

$$\mathcal{A} = e^{-\frac{f(\lambda)}{4}(\log \mu)^2} \quad (3.39)$$

where  $\mu$  is a mass scale that is acting as an IR cutoff. We have one factor like (3.39) for each pair of consecutive gluons. The reason that the divergencies exponentiate in this way is the following. The divergencies come from the exchange of soft gluons among two consecutive hard gluons. In the limit that the hard gluon momenta are infinite, the configuration of hard gluons is invariant under two symmetries: boosts and scale transformations. Let us denote by  $\sigma$  the boost parameter and by  $\tau$  the parameter generating scale transformations,  $x^\mu \rightarrow e^\tau x^\mu$ . Thus naively the amplitude will be of the form

$$\mathcal{A}_{\text{div}} = e^{-h(\lambda)\Delta\sigma\Delta\tau} \quad (3.40)$$

where  $\Delta\sigma$  and  $\Delta\tau$  is the range of scale and conformal transformations where the approximation of exact boost and scaling symmetry are valid. (Recall the classical string result (3.6)). This is conceptually similar to the problem of computing the partition function of a system that is invariant under time translations and spatial translations. The answer will

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<sup>15</sup>We have started with a cutoff scale  $\mu$  as done in the field theory analysis of [7], however, one can verify that if we have started with  $\sqrt{2}\mu$  instead, all the terms containing  $\log 2$  in our final result would disappear.

be  $Z = e^{-fLT}$  where  $L$  and  $T$  are two IR cutoffs. In fact, this can be made more explicit by choosing coordinates in  $R^{1+3}$  that lead to a metric which is Weyl equivalent to a metric where these symmetries are explicit<sup>16</sup>. The function  $f$  that appears in (3.39) is proportional to the cusp anomalous dimension for a Wilson loop in the fundamental representation.<sup>17</sup> Of course we also need another scale so that the log in (3.39) makes sense. This is provided by  $-s = (p_1 + p_2)^2 = 2p_1p_2$  where  $p_1$  and  $p_2$  are the momenta of the two lines. The function  $f$  in (3.39) also appears when one computes the dimension of operators of high spin,  $S$ , of the schematic form  $Tr[\Phi\partial^S\Phi]$ , with  $\Phi$  in the adjoint [25]. These operators have twist

$$\Delta - S = f(\lambda) \log S \tag{3.41}$$

This function was computed perturbatively up to four loops in [26, 27], at strong coupling using strings in AdS in [28, 29] and given exactly as a solution of an integral equation in [30] using integrability (see also [31]). Of course, in addition to the leading divergence in (3.39) we can have a subleading divergence involving a single log. Thus, we can introduce a second function  $g$  through

$$\mathcal{A}_{div,s} = \exp \left\{ -\frac{f(\lambda)}{8} \left( \log \frac{\mu^2}{(-s)} \right)^2 - \frac{g(\lambda)}{4} \left( \log \frac{\mu^2}{(-s)} \right) \right\} \tag{3.42}$$

Where we have defined a new function  $g$ . The precise form of  $g$  will depend on the precise definition of the IR regulator since shifting the log in the first term by a constant can affect the form of  $g$ .<sup>18</sup> In other words, changing  $\mu \rightarrow \mu\kappa$  we change

$$g \rightarrow g + 2f \log \kappa \tag{3.43}$$

Let us now review how these divergences appear when we perform dimensional regularization. The double logs in (3.39) will arise if we have a double pole in  $\epsilon$ . Thus, in dimensional regularization the divergences should organize as [7–9]

$$\mathcal{A}_{div,s} = \exp \left\{ -\frac{1}{8\epsilon^2} f^{(-2)} \left( \frac{\lambda_4 \mu^{2\epsilon}}{s^\epsilon} \right) - \frac{1}{4\epsilon} g^{(-1)} \left( \frac{\lambda_4 \mu^{2\epsilon}}{s^\epsilon} \right) \right\} \tag{3.44}$$

This formula, together with (3.24), gives a precise definition for  $g$ . We now see that expanding in  $\epsilon$  we reproduce the double logs in (3.39) if

$$\left( \lambda \frac{d}{d\lambda} \right)^2 f^{(-2)}(\lambda) = f(\lambda), \quad \lambda \frac{d}{d\lambda} g^{(-1)}(\lambda) = g(\lambda) \tag{3.45}$$

<sup>16</sup>One can write the metric of  $R^{1+3}$  as  $ds^2 = e^{2\tau}(-d\tau^2 + ds_{H_3}^2)$ . Since the theory is conformal invariant we can drop the conformal factor in the metric. We can further choose coordinates in hyperbolic space so that we have now the metric  $ds^2 = -d\tau^2 + d\rho^2 + \cosh^2 \rho d\sigma^2 + \sinh^2 \rho d\varphi^2$ . Boosts and scale transformations correspond now to shifts of  $\tau$  and  $\sigma$ .

<sup>17</sup>For two Wilson lines forming a spacelike cusp where the two lines differ by a large boost parameter  $\gamma$  the anomalous dimension is  $\langle W \rangle \sim e^{-\frac{f}{4}\gamma \log(L_{IR}/L_{UV})}$ , where  $L$  are UV and IR cutoffs. We refer the reader to [21, 24, 25] and references therein for a discussion of all these ideas.

<sup>18</sup>What we called  $g$  here is called  $\mathcal{G}_0$  in [7].

By comparing the general expression for the IR divergence (3.44) to our result (3.38) we can compute the functions  $f$  and  $g$  at strong coupling. We find

$$f = 4\frac{\sqrt{\lambda}}{4\pi}, \quad g = \frac{\sqrt{\lambda}}{4\pi}2(1 - \log 2) \quad (3.46)$$

Notice that the square root of  $\lambda$  dependence introduces a factor of 2 when we integrate as in (3.45). Of course, the function  $f$  has the same value that was computed in other ways in [28, 21] this is implied by the general theory of IR divergences we have just reviewed. In appendix C we discuss the extrapolation of the weak coupling results to strong coupling and the comparison with our answer (3.46).

### 3.6 Field theory results for the four point amplitude

The four point scattering amplitude in maximally supersymmetric theories in  $D$  dimensions has the form

$$\mathcal{A} = \mathcal{A}_{\text{Tree}}a(s, t) \quad (3.47)$$

This is completely determined by the fact that the theory has 16 supersymmetries.<sup>19</sup> A simple way to understand it is the following. In a theory with 16 supersymmetries the generic massive representation has  $2^8$  components. The gluons live in a representation which has  $16 = 2^4$  components. This is possible because it is a massless representation and half of the supercharges act trivially. When we think about a  $2 \rightarrow 2$  scattering process we can see that the state formed by two gluons transforms as a massive representation and has  $2^8$  states, which is the same as the total number of polarization states for two gluons. This means that there is a unique intermediate state. This fixes the  $S$  matrix uniquely up to a common function.<sup>20</sup> Of course this  $S$  matrix determined by the symmetries will be equal to the tree level S-matrix which preserves all symmetries. This explains the form of the four point scattering amplitude in (3.47) in all dimensions. In four dimensions the function  $a(s, t)$  has to obey further constraints. If we demand scaling symmetry we would conclude that it is a function of  $a(s/t)$  and if we further demand that it is invariant under special conformal symmetries we would conclude that  $a$  is a constant. However, due to the IR divergencies we have an additional dependence on the cutoff, which enables the scattering amplitude to be a non-constant function of  $s, t$ .<sup>21</sup>

Bern, Dixon and Smirnov [7] made a conjecture for the exact form of the four loop amplitude of the form

$$\mathcal{A} = \mathcal{A}_{\text{tree}}(\mathcal{A}_{\text{div},s})^2(\mathcal{A}_{\text{div},t})^2 \exp \left\{ \frac{f(\lambda)}{8} \left[ \left( \log \frac{s}{t} \right)^2 + 4\pi^2/3 \right] + C(\lambda) \right\} \quad (3.48)$$

where  $C(\lambda)$  is only a function of the coupling,  $f$  is the same function appearing above (3.39), and  $\mathcal{A}_{\text{div},s}$  is given by (3.44). We see that the momentum dependent finite piece of our

<sup>19</sup>See for instance [32–34] and appendix E of [35].

<sup>20</sup>A similar argument was used in [36] for  $S$  matrices on spin chains.

<sup>21</sup>It seems possible that the functional form of the IR regularized result is still determined by the action of the special conformal generators, though we were not able to prove it.

strong coupling expression has precisely the form predicted by [7], including all numerical factors after we include the appropriate strong coupling form for the cusp anomalous dimension (3.46). Unfortunately we cannot test [7] for the the constant finite pieces, though we could do it if we computed the  $n$  point amplitudes, which are also predicted in [7]. Of course, what we can definitely say is that (3.37) implies a strong coupling value for a combination of the constants introduced in [7].

## 4. Conclusions

In this paper we have given a prescription for computing planar gluon scattering amplitudes at strong coupling using the gauge theory/gravity duality. The computation involves finding a classical string solution moving in (complexified)  $AdS_5$  with a prescribed asymptotic behavior determined by the gluon momenta. The computation can be regularized using the gravity version of dimensional regularization. Though our prescription would work for an arbitrary  $n$  point amplitude, we have only computed explicit answers for the four point amplitude. We found results that agree in all detail with the conjecture of Bern, Dixon and Smirnov [7]. The structure of IR divergences is precisely as expected from general field theory reasoning. The detailed comparison enables us to extract the strong coupling form of the function that determines the subleading divergent terms. The leading divergent terms are determined by the cusp anomalous dimension which is already known at strong coupling [28, 21].

It is amusing that the computation is mathematically similar to the computation we would do in order to compute the expectation values of locally lightlike Wilson loops. This formal similarity, however, might disappear at higher orders in  $1/\sqrt{\lambda}$ .

It seems possible to try to check the conjecture of [7] for  $n$  point functions for  $n > 4$ . Since the  $AdS_5$  sigma model is integrable, it should be possible to find the appropriate classical solutions and evaluate the action. Or even evaluate the action without finding the full explicit form of the classical solutions!

It would also be interesting to learn how to do computations of scattering amplitudes exactly as a function of the coupling by using integrability and spin chains. For this we note that performing computations on  $S^3 \times R$  would be another way to impose an IR cutoff. In that situation it would seem that a single gluon would correspond to a configuration containing spiky strings, similar in spirit to the ones in [37]. It might be possible to consider the scattering of spikes by relating them to some objects on the spin chain.

Hopefully, these amplitudes might be helpful for understanding aspects of QCD and the transition between the perturbative and non-perturbative regimes.

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### A. Solution in the original global coordinates

We start by considering the solution in terms of the original Poincare coordinates (we set  $R = 1$  as it doesn't play any role in the following discussion.)

$$\begin{aligned}
 x_0 &= \frac{i}{2a} \sqrt{1+b^2} (\cosh^2 u_2 - \cosh^2 u_1) , \\
 z &= \frac{1}{a} (\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2) , \\
 x_1 &= \frac{i}{a} \left( \frac{u_2}{2} + \frac{1}{4} \sinh 2u_2 + b \left( -\frac{u_1}{2} + \frac{1}{4} \sinh 2u_1 \right) \right) , \\
 x_2 &= \frac{i}{a} \left( -\frac{u_1}{2} - \frac{1}{4} \sinh 2u_1 + b \left( \frac{u_2}{2} - \frac{1}{4} \sinh 2u_2 \right) \right)
 \end{aligned} \tag{A.1}$$

In order to study the intersection of our surface with the boundary of AdS we write the solution in term of embedding coordinates

$$\begin{aligned}
 X^\mu &= \frac{x^\mu}{z}, \quad \mu = 0, \dots, 3 \\
 X_{-1} + X_4 &= \frac{1}{z}, \\
 X_{-1} - X_4 &= \frac{z^2 + x_\mu x^\mu}{z}
 \end{aligned} \tag{A.2}$$

The boundary of AdS in global coordinates is then parametrized by the space of coordinates satisfying

$$-X_{-1}^2 - X_0^2 + X_1^2 + \dots + X_4^2 = 0 \tag{A.3}$$

quotiented by overall rescalings, but they cannot all be zero. The geometry of the boundary is  $R \times S^3$ .

As discussed in the main text, we impose boundary conditions at  $z = \infty$ . Notice that  $z = \infty$  implies the equation  $X_{-1} + X_4 = 0$  which gives a lightlike surface on the bulk that intersects the boundary also on a lightlike surface. The region on the boundary that is bounded by this lightlike surface is conformal to Minkowski space. In fact, that region corresponds to the Minkowski space parametrized by  $x^\mu$  at  $z = 0$ . However, the  $z = \infty$  surface also extends in the interior of  $AdS_5$ . In order to see whether the region  $u_i \rightarrow \pm\infty$  is on the boundary or not we need to check the behavior of the  $x^\mu$  coordinates. Without loss of generality lets assume  $u_1 \gg u_2 \sim 1$ . In this regime we obtain the following expression for the embedding coordinates<sup>22</sup>

$$\begin{aligned}
 X_0 &= -\frac{i\sqrt{1+b^2}}{4} \frac{e^{u_1}}{\cosh u_2 + b \sinh u_2}, & X_1 &= \frac{ib}{4} \frac{e^{u_1}}{\cosh u_2 + b \sinh u_2}, & X_3 &= 0 \\
 X_2 &= -\frac{i}{4} \frac{e^{u_1}}{\cosh u_2 + b \sinh u_2}, & X_{-1} = -X_4 &= \frac{u_1(b^2 - 1)e^{u_1}}{8a(\cosh u_2 + b \sinh u_2)}
 \end{aligned} \tag{A.4}$$

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<sup>22</sup>We thank Shesansu Pal for pointing out an error in the original version of these equations.

These coordinates are complex but large. Thus we can say that they live in the complexified boundary, which is defined by (A.3) but now the coordinate are complex and we allow complex rescalings.

### B. A useful integral

In this appendix we show how to compute the following integral

$$S = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dudv (\cosh u \cosh v + b \sinh u \sinh v)^\epsilon \tag{B.1}$$

This integral can be done by expanding the integrand as a power series on  $b$ , then integrating term by term and finally performing the sum

$$S = \sum_{l=0}^{\infty} \frac{\Gamma[\epsilon + 1]}{\Gamma[\epsilon + 1 - l]l!} b^{2l} \left( \int du (\cosh u)^\epsilon (\tanh u)^l \right)^2 \tag{B.2}$$

The double integral becomes a sum of integrals on a single variable

$$\int_{-\infty}^{\infty} du (\cosh u)^\epsilon (\tanh u)^l = \frac{(1 + (-1)^l) \Gamma[\frac{1+l}{2}] \Gamma[-\frac{\epsilon}{2}]}{2 \Gamma[\frac{1+l-\epsilon}{2}]} \tag{B.3}$$

This identity is valid when  $\epsilon < 0$ . Finally, performing the sum we obtain

$$S = \frac{\pi \Gamma[-\frac{\epsilon}{2}]^2}{\Gamma[\frac{1-\epsilon}{2}]^2} {}_2F_1 \left( \frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2 \right) \tag{B.4}$$

This expression is valid for all values of  $\epsilon < 0$ . However, we are interested in the behavior for small  $\epsilon$ . We find that

$$F \equiv {}_2F_1 \left( \frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2 \right) = 1 + \frac{1}{2} \log(1 - b^2) \epsilon + \frac{1}{2} \log(1 - b) \log(1 + b) \epsilon^2 + \mathcal{O}(\epsilon^3) \tag{B.5}$$

We find it convenient to express this expansion as follows

$$\begin{aligned} F &= F_s + F_t - \frac{\epsilon^2}{4} \left( \log \frac{1+b}{1-b} \right)^2 \\ F_s &= \frac{1}{2} + \frac{\epsilon}{2} \log(1+b) + \frac{\epsilon^2}{4} (\log(1+b))^2 = \frac{1}{2} (1+b)^\epsilon, \\ F_t &= \frac{1}{2} (1-b)^\epsilon \end{aligned} \tag{B.6}$$

Where all the equalities are valid up to order  $\epsilon^2$ .

### C. Extrapolating the weak coupling results for $g(\lambda)$ to strong coupling

In this appendix we give a rough estimate for  $g(\lambda)$  at strong coupling by using its weak coupling expansion<sup>23</sup>. We follow closely the idea proposed in [38] in order to obtain an

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<sup>23</sup>This calculation was suggested to us by Z. Bern, L. Dixon and R. Roiban, who also did it.

approximate expression for the cusp anomalous dimension valid at all values of the coupling constant.

The weak coupling expansion for the cusp anomalous dimension is [39]

$$f(\lambda) = 8 \left( \frac{\lambda}{16\pi^2} \right) - 16\zeta_2 \left( \frac{\lambda}{16\pi^2} \right)^2 + 176\zeta_4 \left( \frac{\lambda}{16\pi^2} \right)^3 + \dots \quad (\text{C.1})$$

This perturbative information together with its strong coupling behavior  $f(\lambda) \sim \sqrt{\lambda}$  suggest a relation of the form

$$\left( \frac{\lambda}{16\pi^2} \right)^n = \sum_{r=n}^{2n} C_r (f(\lambda))^r \quad (\text{C.2})$$

For a given value of  $n$ , once we fix the coefficients  $C_r$  by using the perturbative data, (C.2) predicts a given strong coupling limit.<sup>24</sup>

At weak coupling  $g(\lambda)$  was computed up to three loops [7]

$$g(\lambda) = -4\zeta_3 \left( \frac{\lambda}{16\pi^2} \right)^2 + 8(4\zeta_5 + 10/3\zeta_2\zeta_3) \left( \frac{\lambda}{16\pi^2} \right)^3 + \dots \quad (\text{C.3})$$

To repeat the analysis discussed above in order to extract the strong coupling behavior of  $g(\lambda)$  is subtle due to two features. First,  $g(\lambda)$  starts at two loops, so the relation (C.2) should be modified accordingly. Second, note that if our prediction is correct,  $g(\lambda)$  should change its sign in going from weak to strong coupling. This does not occur for  $f$ .

A possibility is to define a new function  $\tilde{g}(\lambda)$ , differing from  $g(\lambda)$  by a constant times  $f(\lambda)$ . As seen in the main text, this shift corresponds simply to a change in the IR regulator.

$$\tilde{g}(\lambda) = g(\lambda) + \xi f(\lambda) \quad (\text{C.4})$$

For positive  $\xi$ ,  $\tilde{g}(\lambda)$  will be positive both at weak and strong coupling. It is then straightforward to repeat the analysis of [38] for  $n = 3$  (since we have three coefficients at our disposal). The value obtained at strong coupling should be then compared with our prediction

$$\tilde{g}(\lambda) = (2(1 - \log(2)) + 4\xi) \frac{\sqrt{\lambda}}{4\pi} + \dots \quad (\text{C.5})$$

The following figure shows a comparison between the value predicted by our computation (C.5) (solid blue line), and the value obtained by a naive extrapolation as explained above (dashed red line), for the range  $0 < \xi < 2$

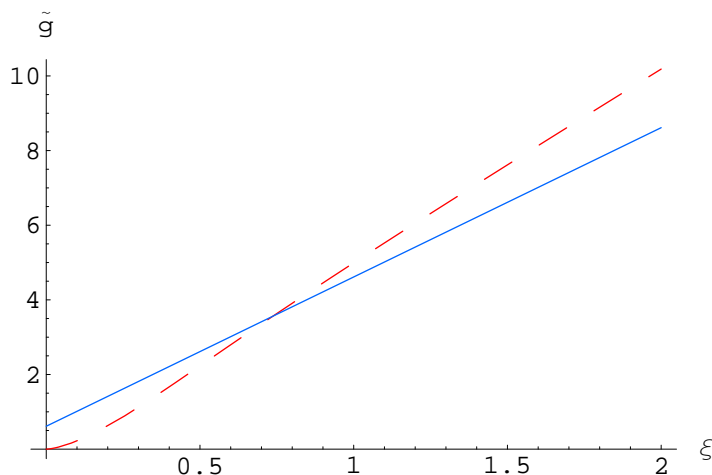
Note that for  $\xi = 0$  the assumption (C.2) is not valid. At the particular value  $\xi = \log(2)/2$  the strong coupling prediction is particularly simple<sup>25</sup>

$$\tilde{g}(\lambda) = 2 \frac{\sqrt{\lambda}}{4\pi} + \dots \quad (\text{C.6})$$

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<sup>24</sup>Actually, see discussion in [26], if one introduces enough perturbative information, namely  $n \geq 4$ , the value predicted for  $f(\lambda)$  at strong coupling is in very good agreement with the known strong coupling value. Furthermore, combining the known weak and strong coupling information one can get an answer accurate to 1 % (when compared against the answer from the BES equation) for the whole range of the coupling constant.

<sup>25</sup>Note that if  $\log(2)$  has transcendentality 1, then such a shift doesn't break transcendentality.



**Figure 6:** Comparison between the result for  $\tilde{g}$  obtained from our computation (solid blue line) and a naive extrapolation from weak coupling (dashed red line).

whereas the value from the extrapolation turns out to be

$$\tilde{g}(\lambda) \approx 1.37 \frac{\sqrt{\lambda}}{4\pi} + \dots \tag{C.7}$$

This differs by a 30% from the predicted number. All together, given the little that is known for the function  $g(\lambda)$ , all that we can say is that a naive extrapolation from weak to strong coupling seems to be compatible with our results.

Of course, the above analysis is by no means rigorous. In order to make a more precise comparison one would like to have the analog of the BES equation from which  $g(\lambda)$  could be computed at any order, then one could proceed along the lines of [40, 41].

## References

- [1] G. Sterman and S. Weinberg, *Jets from quantum chromodynamics*, *Phys. Rev. Lett.* **39** (1977) 1436.
- [2] CTEQ collaboration, R. Brock et al., *Handbook of perturbative QCD: version 1.0*, *Rev. Mod. Phys.* **67** (1995) 157.
- [3] R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and collider physics*, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **8** (1996) 1.
- [4] Z. Bern, L.J. Dixon and D.A. Kosower, *Three-loop superfiniteness of  $N = 8$  supergravity*, *Comptes Rendus Physique* **5** (2004) 955 [[hep-th/0410021](#)].
- [5] J.M. Maldacena, *The large- $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [6] D.J. Gross and P.F. Mende, *The high-energy behavior of string scattering amplitudes*, *Phys. Lett.* **B 197** (1987) 129; *String theory beyond the Planck scale*, *Nucl. Phys.* **B 303** (1988) 407.
- [7] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [[hep-th/0505205](#)].



- [8] L. Magnea and G. Sterman, *Analytic continuation of the sudakov form-factor in QCD*, *Phys. Rev. D* **42** (1990) 4222.
- [9] G. Sterman and M.E. Tejeda-Yeomans, *Multi-loop amplitudes and resummation*, *Phys. Lett. B* **552** (2003) 48 [[hep-ph/0210130](#)].
- [10] S. Catani, *The singular behaviour of QCD amplitudes at two-loop order*, *Phys. Lett. B* **427** (1998) 161 [[hep-ph/9802439](#)].
- [11] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [[hep-th/0309040](#)].
- [12] M. Rho, S.-J. Sin and I. Zahed, *Elastic parton parton scattering from AdS/CFT*, *Phys. Lett. B* **466** (1999) 199 [[hep-th/9907126](#)].
- [13] R.A. Janik and R. Peschanski, *High energy scattering and the AdS/CFT correspondence*, *Nucl. Phys. B* **565** (2000) 193 [[hep-th/9907177](#)].
- [14] J. Polchinski and M.J. Strassler, *Hard scattering and gauge/string duality*, *Phys. Rev. Lett.* **88** (2002) 031601 [[hep-th/0109174](#)]. J. Polchinski and M.J. Strassler, *Deep inelastic scattering and gauge/string duality*, *JHEP* **05** (2003) 012 [[hep-th/0209211](#)].
- [15] S.J. Brodsky and G.F. de Teramond, *Light-front hadron dynamics and AdS/CFT correspondence*, *Phys. Lett. B* **582** (2004) 211 [[hep-th/0310227](#)].
- [16] O. Andreev, *A few comments on the high-energy behavior of string scattering amplitudes in warped spacetimes*, *Phys. Rev. D* **70** (2004) 027901 [[hep-th/0402017](#)].
- [17] R.C. Brower, J. Polchinski, M.J. Strassler and C.I. Tan, *The Pomeron and gauge/string duality*, [hep-th/0603115](#).
- [18] R. Kallosh and A.A. Tseytlin, *Simplifying superstring action on  $AdS_5 \times S^5$* , *JHEP* **10** (1998) 016 [[hep-th/9808088](#)].
- [19] J.M. Maldacena, *Wilson loops in large- $N$  field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)]. S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large- $N$  gauge theory and Anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [[hep-th/9803001](#)].
- [20] D.J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, *Adv. Theor. Math. Phys.* **4** (2000) 893 [[hep-th/0008075](#)].
- [21] M. Kruczenski, *A note on twist two operators in  $N = 4$  SYM and Wilson loops in Minkowski signature*, *JHEP* **12** (2002) 024 [[hep-th/0210115](#)].
- [22] W. Siegel, *Supersymmetric dimensional regularization via dimensional reduction*, *Phys. Lett. B* **84** (1979) 193.
- [23] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large- $N$  limit of theories with sixteen supercharges*, *Phys. Rev. D* **58** (1998) 046004 [[hep-th/9802042](#)].
- [24] J.C. Collins, *Sudakov form factors*, *Adv. Ser. Direct. High Energy Phys.* **5** (1989) 573 [[hep-ph/0312336](#)].
- [25] G.P. Korchemsky, *Asymptotics of the Altarelli-Parisi-Lipatov evolution kernels of parton distributions*, *Mod. Phys. Lett. A* **4** (1989) 1257.

- [26] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *On-shell methods in perturbative QCD*, hep-th/0610248.
- [27] F. Cachazo, M. Spradlin and A. Volovich, *Four-loop cusp anomalous dimension from obstructions*, hep-th/0612309.
- [28] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [hep-th/0204051].
- [29] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in  $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [hep-th/0204226].
- [30] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* **0701** (2007) 21 [hep-th/0610251].
- [31] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for  $AdS_5 \times S^5$  strings*, *JHEP* **11** (2006) 070 [hep-th/0609044];  
B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* **11** (2006) 014 [hep-th/0603157].
- [32] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, *Supergravity and the s matrix*, *Phys. Rev. D* **15** (1977) 996.
- [33] M.T. Grisaru and H.N. Pendleton, *Some properties of scattering amplitudes in supersymmetric theories*, *Nucl. Phys. B* **124** (1977) 81.
- [34] S.J. Parke and T.R. Taylor, *Perturbative QCD utilizing extended supersymmetry*, *Phys. Lett. B* **157** (1985) 81.
- [35] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys. B* **530** (1998) 401 [hep-th/9802162].
- [36] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended  $SU(2|2)$  symmetry*, *J. Stat. Mech.* **01** (2007) 017 [nlin.si/0610017].
- [37] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, *JHEP* **08** (2005) 014 [hep-th/0410226].
- [38] A.V. Kotikov, L.N. Lipatov and V.N. Velizhanin, *Anomalous dimensions of wilson operators in  $N = 4$  SYM theory*, *Phys. Lett. B* **557** (2003) 114 [hep-ph/0301021].
- [39] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko and V.N. Velizhanin, *Three-loop universal anomalous dimension of the Wilson operators in  $N = 4$  SUSY Yang-Mills model*, *Phys. Lett. B* **595** (2004) 521 [Erratum *ibid.* **595** (2004) 521] [hep-th/0404092].
- [40] M.K. Benna, S. Benvenuti, I.R. Klebanov and A. Scardicchio, *A test of the AdS/CFT correspondence using high-spin operators*, *Phys. Rev. Lett.* **98** (2007) 131603 [hep-th/0611135].
- [41] L.F. Alday, G. Arutyunov, M.K. Benna, B. Eden and I.R. Klebanov, *On the strong coupling scaling dimension of high spin operators*, hep-th/0702028.